30. Cross-correlation & Autocorrelation

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$$R_{xy}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, x(t)\, dt$$

$$|R_{xx}(\tau)| \le R_{xx}(0)$$

The **cross-correlation** of two signals $x(t)$ and $y(t)$ is denoted by $R_{xy}(\tau)$ or $(x \star y)(\tau)$ and is defined as

$$R_{xy}(\tau) = (x \star y)(\tau) = \int_{-\infty}^{\infty} x^*(t)\, y(t+\tau)\, dt = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

Cross-correlation is quite similar to convolution,

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

The difference is that you shift $x^*(t)$ but without the need to create the time-inverted version $x^*(-t)$, just use $x^*(t)$, move the anchor point to τ, evaluate the multiplication $x^*(t-\tau)\, y(t)$, then find the area under such multiplication.

For periodic signals $x(t)$ and $y(t)$, with same fundamental period T_0:

$$R_{xy}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t)\, y(t+\tau)\, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-\tau)\, y(t)\, dt$$

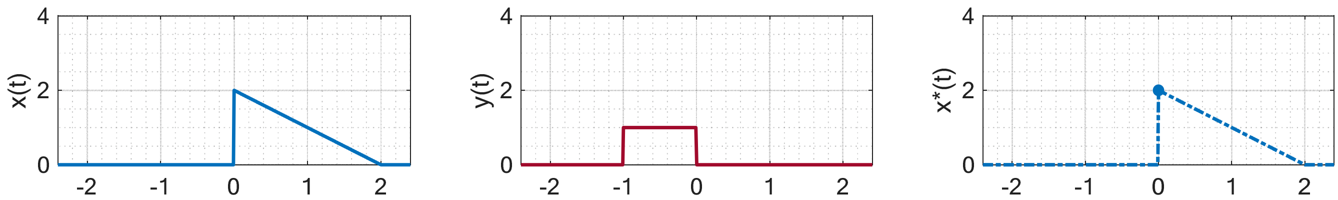
Cross-correlation measures the **similarity** between the two signals $x(t)$ and $y(t)$ at different time shifts τ. Cross-correlation can be used to find short known features inside a longer signal. It has applications in pattern recognition, signal detection, averaging, cryptanalysis, laser microscopy, etc.

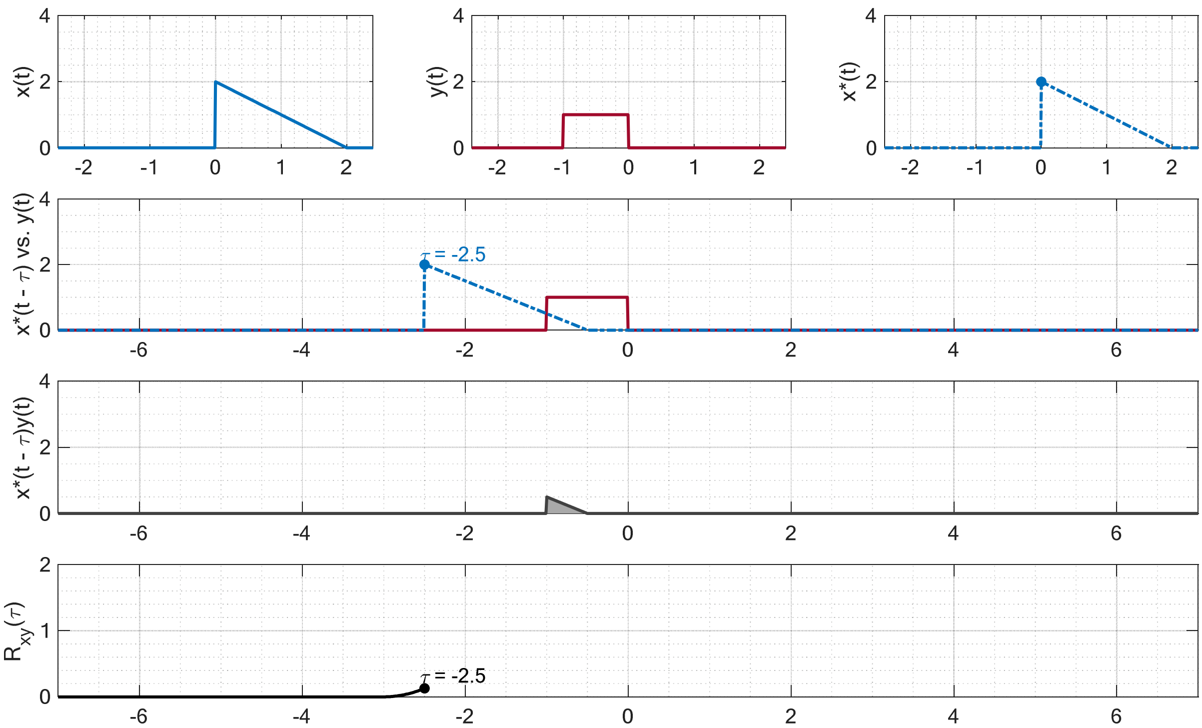
We can easily see that

$$R_{xy}(\tau) = R_{y^*x^*}(-\tau)$$

Q1. For the signals $x(t) = 2\,\Delta(t/2)u(t)$ and $y(t) = \mathrm{rect}(t+0.5)$, determine the cross-correlation function $R_{xy}(\tau)$.

Q1. Solution. To perform the graphical solution, first draw $x(t)$ and $y(t)$. There is NO need to create $y(-t)$, rather use $x^*(t) = x(t)$.





Region #1: For time-shift $\tau < -3$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $-3 \leq \tau < -2$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xy}(\tau) = \int_{-1}^{\tau+2} \big(2 - (t-\tau)\big) \times 1\, dt$$

$$R_{xy}(\tau) = \int_{-1}^{\tau+2} (2+\tau)\, dt - \int_{-1}^{\tau+2} t\, dt$$

$$R_{xy}(\tau) = (2+\tau) \times [t]_{-1}^{\tau+2} - \left[\frac{t^2}{2}\right]_{-1}^{\tau+2}$$

$$R_{xy}(\tau) = (2+\tau) \times (\tau+3) - \left[\frac{(\tau+2)^2 - (-1)^2}{2}\right]$$

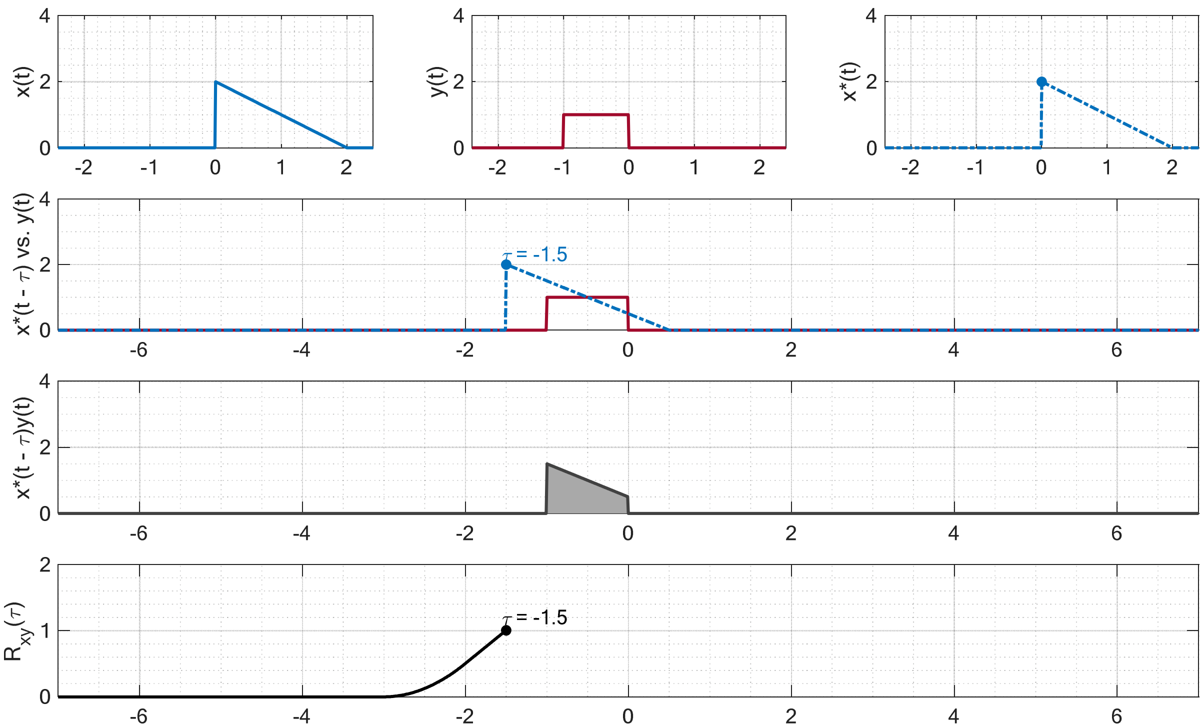
$$R_{xy}(\tau) = \tau^2 + 5\tau + 6 - \left[\frac{\tau^2 + 4\tau + 4 - 1}{2}\right] = \frac{\tau^2 + 6\tau + 9}{2} = \frac{(\tau+3)^2}{2}$$

Notice here that we have a triangle with width of $\tau + 2 - (-1) = \tau + 3$ and similar height (due to the slope of -1). This gives us a quicker solution using the triangle area:

$$R_{xy}(\tau) = \frac{1}{2} \times W \times H = \frac{1}{2} \times (\tau+3) \times (\tau+3)$$

$$R_{xy}(\tau) = \frac{(\tau+3)^2}{2}$$

$$R_{xy}(\tau) = \frac{\tau^2 + 6\tau + 9}{2}$$



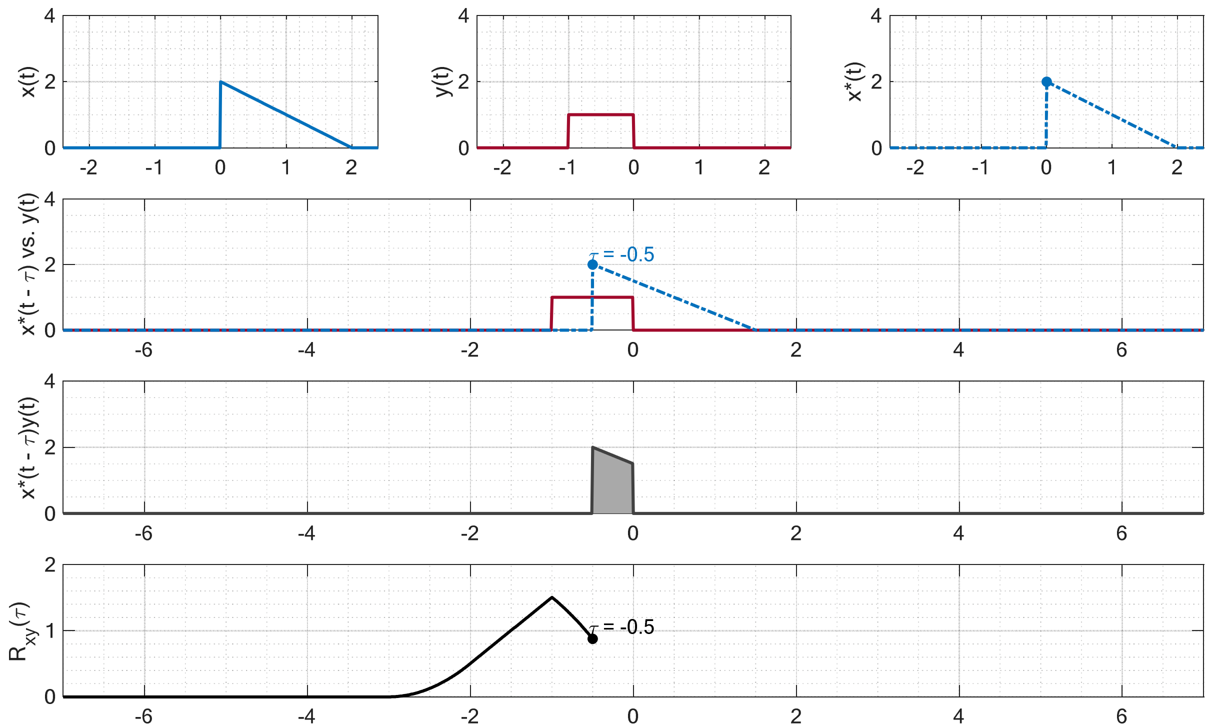
Region #3: For time-shift $-2 \le \tau < -1$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xy}(\tau) = \int_{-1}^{0} \big(2 - (t-\tau)\big) \times 1\, dt = \frac{2\tau + 5}{2}$$

Note: This is a rotated trapezoid with height of $0 - (-1) = 1$ and two bases of length $\tau + 2 - 0$ and $\tau + 2 - (-1)$, which we figured out from the slope of -1. Hence, the trapezoid area (which is the desired integral):

$$R_{xy}(\tau) = \left(\frac{B_1 + B_2}{2}\right) \times H = \frac{(\tau + 2) + (\tau + 3)}{2} \times 1 = \frac{2\tau + 5}{2}$$



Region #4: For time-shift $-1 \le \tau < 0$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xy}(\tau) = \int_{\tau}^{0} \big(2-(t-\tau)\big) \times 1\, dt = \left(\frac{\tau+4}{2}\right)(-\tau)$$

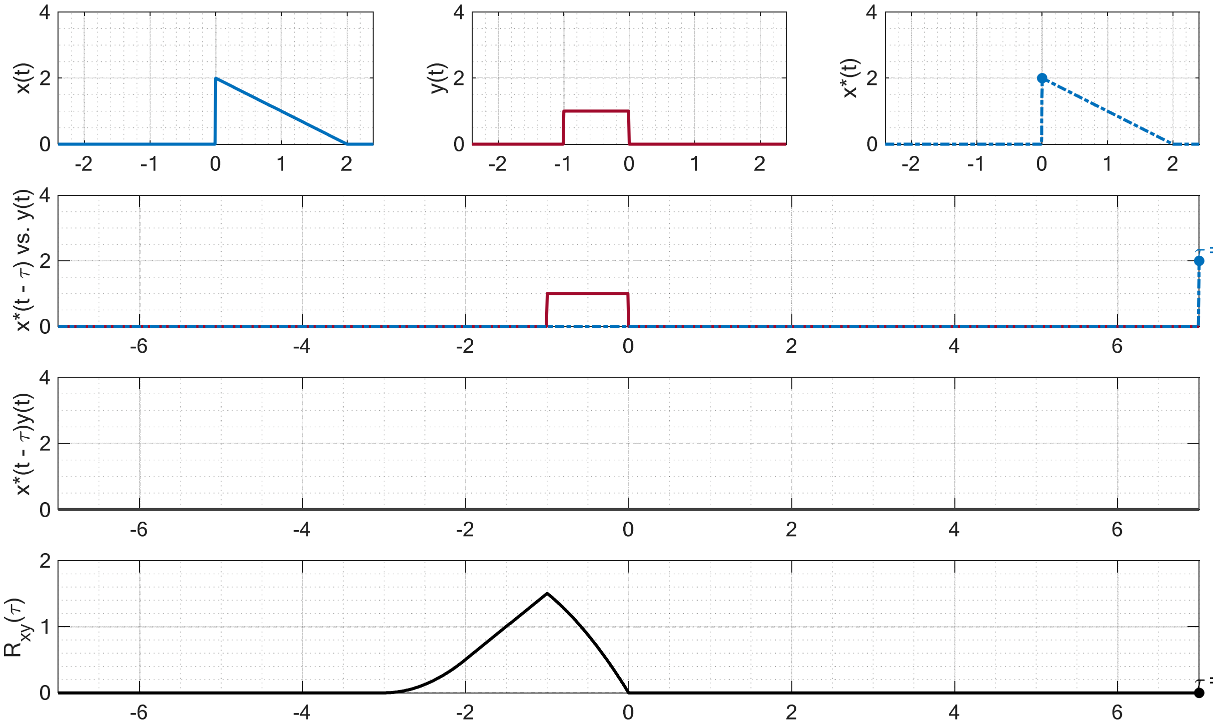
Note: This is a rotated trapezoid with height of $0-\tau$ and two bases of length $\tau+2-0$ and 2, which we found because the slope is -1. Hence, the trapezoid area (which is the desired integral):

$$R_{xy}(\tau) = \left(\frac{B_1+B_2}{2}\right) \times H = \frac{(\tau+2)+(2)}{2} \times (-\tau) = \left(\frac{\tau+4}{2}\right)(-\tau)$$

Region #5: For time-shift $\tau \geq 0$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, y(t)\, dt$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$



Full Solution (five regions):

$$R_{xy}(\tau) = \begin{cases} 0, & \tau < -3 \\ \dfrac{(\tau+3)^2}{2}, & -3 \le \tau < -2 \\ \dfrac{2\tau+5}{2}, & -2 \le \tau < -1 \\ \left(\dfrac{\tau+4}{2}\right)(-\tau), & -1 \le \tau < 0 \\ 0, & \tau \ge 0 \end{cases}$$

The **autocorrelation** $R_{xx}(\tau)$ of a signal $x(t)$ is defined as

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt = \int_{-\infty}^{\infty} x^*(t-\tau)\, x(t)\, dt$$

which is the cross-correlation of $x(t)$ with its own self.

For periodic signal $x(t)$, with fundamental period T_0, integration from $-\infty$ to ∞ is replaced by integration over one period and we divide by the period (similar to average value calculation):

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t)\, x(t+\tau)\, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-\tau)\, x(t)\, dt$$

Autocorrelation measures how similar the signal $x(t)$ to a delayed copy of itself, for each amount of time delay τ. It has applications in finding repeating patterns, such as identifying periodic signal obscured by noise, or recognizing the fundamental frequency in a signal affected by other harmonic frequencies. Autocorrelation can also be used to easily calculate the power spectral density.

Some $R_{xx}(\tau)$ properties:

$R_{xx}(-\tau) = R_{xx}(\tau)$, when signal $x(t)$ is real-valued [even symmetry].

$R_{xx}(-\tau) = R_{xx}^*(\tau)$, when signal $x(t)$ is complex-valued.

$|R_{xx}(\tau)| \leq R_{xx}(0)$, maximum value of $R_{xx}(\tau)$ always at $\tau = 0$, and real

$R_{xx}(0) = P_x$ for periodic $x(t)$ and $R_{xx}(0) = E_x$ for aperiodic $x(t)$.

Proof. For periodic $x(t)$ that is a power signal,

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-\tau)\, x(t)\, dt$$

$$R_{xx}(0) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-0)\, x(t)\, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = P_x$$

For aperiodic $x(t)$ that is an energy signal,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t-\tau)\, x(t)\, dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x^*(t-0)\, x(t)\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt = E_x$$

Q2. Determine autocorrelation $R_{xx}(\tau)$ for the signal $x(t) = A\cos(\omega_0 t)$.

Q2. Solution. Substitute in the integral definition of autocorrelation for a real-valued periodic signal

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^*(t-\tau)\, x(t)\, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t-\tau)\, x(t)\, dt$$

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} A\cos\big(\omega_0(t-\tau)\big)\, A\cos(\omega_0 t)\; dt$$
$$= \frac{A^2}{T_0}\int_{t_0}^{t_0+T_0} \cos(\omega_0 t - \omega_0\tau)\cos(\omega_0 t)\; dt$$

$$R_{xx}(\tau) = \frac{A^2}{T_0}\int_{t_0}^{t_0+T_0} \cos(\omega_0 t - \omega_0\tau)\cos(\omega_0 t)\; dt$$
$$= \frac{A^2}{T_0}\int_{t_0}^{t_0+T_0} \left[\frac{1}{2}\cos(2\omega_0 t - \omega_0\tau) + \frac{1}{2}\cos(\omega_0\tau)\right] dt$$
$$= \frac{A^2}{2T_0}\int_{t_0}^{t_0+T_0} \cos(2\omega_0 t - \omega_0\tau)\, dt + \frac{A^2}{2T_0}\int_{t_0}^{t_0+T_0} \cos(\omega_0\tau)\, dt$$
$$= 0 + \frac{A^2}{2T_0}\cos(\omega_0\tau)\int_{t_0}^{t_0+T_0} 1\, dt$$

$$R_{xx}(\tau) = \frac{A^2}{2T_0}\cos(\omega_0\tau)\,[t]_{t_0}^{t_0+T_0} = \frac{A^2}{2}\cos(\omega_0\tau)$$